

- Suppose you have a square matrix  $A$ . Let  $A$  represent a linear transformation that is multiplied by a vector  $V$  (i.e. let  $A$  be a transformation matrix upon another vector  $V$ ).
  - If  $AV$  is parallel to  $V$  (i.e. the transformed vector is parallel to the original vector), then  $V$  is an **eigenvector** of  $A$ .
    - By convention,  $V \neq \vec{0}$ .
  - Consequently, the constant factor by which the magnitude of the vector has changed is the **eigenvalue** associated with  $V$  and  $A$ .
  - i.e.  $AV = \lambda V$ , where  $V$  represents the eigenvector and  $\lambda$  represents the associated eigenvalue.
- **Characteristic polynomial**  $= \det(A - \lambda I)$ 
  - Where does this come from?
    - $AV = \lambda V \Rightarrow AV - \lambda V = \vec{0} \Rightarrow (A - \lambda I)V = \vec{0}$ , where  $I$  is the identity matrix.
    - $(A - \lambda I)V = \vec{0}$  iff  $\det(A - \lambda I) = 0$ .
      - $\det(A - \lambda I)$  is the **characteristic polynomial** of  $A$ .
  - For a two-by-two matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .
    - $\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ab - bc = 0$
    - $\lambda^2 - (a + d)\lambda + ab - bc = 0$ 
      - **Trace**  $T = a + d$
      - **Determinant**  $D = ad - bc$
      - Characteristic equation can be written as  $\lambda^2 - T\lambda + D = 0$ .
- How to find eigenvalues and eigenvectors:
  - 1. Find the characteristic polynomial of  $A$  (i.e. find  $\det(A - \lambda I)$ ).
  - 2. Solve  $\det(A - \lambda I) = 0$  to obtain a set of eigenvalues.
  - 3. For each eigenvalue, find an associated eigenvector by substituting back into the equation  $(A - \lambda I)V = \vec{0}$  and solving the system of equations.
    - The system of equations should be redundant (i.e. each individual equation in the system should be linearly dependent on all the others).
    - Note: Any eigenvector will do, as every eigenvalue associated with a specific eigenvector will just be multiples of each other (i.e. they will have the same direction). But for practical purposes, most people choose the most simplified eigenvector (i.e. choose  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  over  $\begin{pmatrix} 3 \\ -6 \end{pmatrix}$ ).
- Applications
  - Solving systems of differential equations
  - Transforming images (e.g. scaling, rotating, etc.)
  - Vibration analysis
  - Computational chemistry
    - Schrödinger equation
    - Molecular orbital theory